A Parallel Implementation of Three-Dimensional, Lagrangian, Shallow Water Eqauations

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Our work has focused on the f-plane shollow-water system:

$$h_t + (h\bar{\bar{u}})_x + (h\bar{\bar{v}})_y = 0. \tag{1}$$

$$h\left(\bar{\bar{u}}_t + \bar{\bar{u}}\bar{\bar{u}}_x + \bar{\bar{v}}\bar{\bar{u}}_y\right) + \mathcal{C}^2 h\left(a_x + h_x\right) - fh\bar{\bar{v}} = \left(\epsilon h\left(\bar{\bar{u}}_x - \bar{\bar{v}}_y\right)\right)_x + \left(\epsilon h\left(\bar{\bar{u}}_y + \bar{\bar{v}}_x\right)\right)_y - \frac{\epsilon k_1 h\bar{\bar{u}}}{\lambda^2} + \frac{\epsilon k_2}{\lambda^2}\left(u_w - \bar{\bar{u}}\right),$$

and, (2)

$$h\left(\bar{\bar{v}}_t + \bar{\bar{u}}\bar{\bar{v}}_x + \bar{\bar{v}}\bar{\bar{v}}_y\right) + \mathcal{C}^2 h\left(a_y + h_y\right) + fh\bar{\bar{u}} = \left(\epsilon h\left(\bar{\bar{u}}_y + \bar{\bar{v}}_x\right)\right)_x - \left(\epsilon h\left(\bar{\bar{u}}_x - \bar{\bar{v}}_y\right)\right)_y - \frac{\epsilon k_1 h\bar{\bar{v}}}{\lambda^2} + \frac{\epsilon k_2}{\lambda^2}\left(v_w - \bar{\bar{v}}\right). \tag{3}$$

The fields $(\bar{\bar{u}}, \bar{\bar{v}})(x, y, t)$ are the depth-averaged x-y components of the velocity field; the surface $z = a(x, y)$ represents the bottom topography; the free surface of the fluid is given by $z = a(x, y) + h(x, y, t)$ and the fluid occupies the region where $h(x, y, t) > 0$. We determine the free-boundary where $h(x, y, t) = 0$ as part of the solution.

The turbulent viscous-stresses are given by

$$\Sigma = \epsilon h \begin{pmatrix} \bar{\bar{u}}_x - \bar{\bar{v}}_y, & \bar{\bar{u}}_y + \bar{\bar{v}}_x \\ \bar{\bar{u}}_y + \bar{\bar{v}}_x, & -\left(\bar{\bar{u}}_x - \bar{\bar{v}}_y\right) \end{pmatrix} \tag{4}$$

and this stress-tensor is purposely taken to be deviatoric and thus trace free. The terms $(fh\bar{\bar{v}}, -fh\bar{\bar{u}})$ in (2) and (3) represent the Coriolis forces, the terms

$$\left(\frac{-\epsilon k_1 h\bar{\bar{u}}}{\lambda^2}, \frac{-\epsilon k_1 h\bar{\bar{v}}}{\lambda^2}\right)$$

represent the bottom friction forces, the terms

$$\left(\frac{-\epsilon k_2}{\lambda^2}\left(u_w - \bar{\bar{u}}\right), \frac{-\epsilon k_2}{\lambda^2}\left(v_w - \bar{\bar{v}}\right)\right)$$

represent the wind forces generated by wind velocities (u_w, v_w). Finally $\lambda^2 \ll 1$ represents the ratio $\left(\frac{H}{L}\right)^2$ where H is a typical vertical length scale and L is a typical lateral length scale and $\mathcal{C}^2$ is the inverse Froude number.

Novel features of our work involve a "Lagrangian" reformulation of (1)-(3) and a robust Lagrangian code to integrate the system (1)-(3). We have developed all codes in MATLAB and have a parallel impelementation of the code for the two-dimensional problem. These codes successfully reproduce the exact solutions obtained when the bottom surface is a paraboloid.